

On the classification of simple modules for
 cyclotomic Hecke algebras of type $G(m, 1, n)$
 and
 Kleshchev multipartitions

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Abstract

We give a proof of a conjecture that Kleshchev multipartitions are those partitions which parametrize non-zero simple modules obtained as factor modules of Specht modules by their own radicals.

1 Introduction

After Hecke algebras appeared, unexpectedly deep applications and results have been found in the representation theory of these algebras. Concerned with ordinary representations, Lusztig's cell theory is the main driving force. But we do not consider it here. The other interest is about the modular representation theory of these algebras. We are mainly working with Hecke algebras of type A and type B , and this research is driven by Dipper and James [DJ1][DJ2]. Recently, a new type of Hecke algebras was introduced. We call them cyclotomic Hecke algebras of type $G(m, 1, n)$ following [BM]. Hecke algebras of type A and type B are special cases of these algebras. The author studied modular representations over the algebra for the case that parameters were roots of unity in the field of complex numbers [A1]. In particular, it gives the classification of simple modules. The removal of the restriction on base fields was achieved in [AM]. In the paper [AM], we gave a classification of the simple modules over cyclotomic Hecke algebras in terms of the crystal graphs of integrable highest weight modules over certain quantum algebras. The result turns out to be useful for verifying a conjecture of Vigneras [Vig3].

On the other hand, another approach was already proposed in [GL][DJM]. Main results in the theory are that we can define "Specht modules", and that each Specht module S^λ has natural bilinear form, and each of $D^\lambda := S^\lambda / \text{rad}S^\lambda$

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is an absolutely irreducible or zero module. Further, the theory claims that the set of non-zero D^λ is a complete set of simple modules.

But there is one drawback. The theory does not tell which D^λ are actually non-zero. We conjectured in [AM] that the crystal graph description gave the criterion. Namely, we conjectured that $D^\lambda \neq 0$ if and only if $\underline{\lambda}$ is a Kleshchev multipartition. The purpose of this paper is to prove the conjecture. It is achieved by interpreting the conjecture into a problem about canonical bases in Fock spaces. This part is based on [A1] and [AM]. Then it is easily verified by using a recent result of Uglov [U].

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2 Preliminaries

Let R be an integral domain, u_1, \dots, u_m be elements in R , and ζ be an invertible element. The Hecke algebra of type $G(m, 1, n)$ is the algebra associated with these parameters is the R -algebra defined by the following defining relations for generators a_i ($1 \leq i \leq n$). We denote this algebra by \mathcal{H}_n .

$$(a_1 - u_1) \cdots (a_1 - u_m) = 0, \quad (a_i - \zeta)(a_i + \zeta^{-1}) = 0 \quad (i \geq 2)$$

$$a_1 a_2 a_1 a_2 = a_2 a_1 a_2 a_1, \quad a_i a_j = a_j a_i \quad (j \geq i+2)$$

$$a_i a_{i-1} a_i = a_{i-1} a_i a_{i-1} \quad (3 \leq i \leq n)$$

It is known that this algebra is R -free of rank $m^n n!$ as an R -module. This algebra is also known to be cellular in the sense of Graham and Lehrer [GL], and thus has Specht modules. Following [DJM], we shall explain the theory. A **partition** λ of size n is a sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \dots$ such that $\sum \lambda_i = n$. We write $|\lambda| = n$. A **multipartition** of size n is a sequence of m partitions $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ such that $\sum_{k=1}^m |\lambda^{(k)}| = n$. If $n = 0$, we denote the multipartition by \emptyset . The set of multipartitions has a poset structure. The partial order is the **dominance order**, which is defined as follows.

Definition 2.1 *Let $\underline{\lambda}$ and $\underline{\mu}$ be multipartitions. We say that $\underline{\lambda}$ dominates $\underline{\mu}$, and write $\underline{\lambda} \triangleright \underline{\mu}$ if we have*

$$\sum_{l=1}^{k-1} |\lambda^{(l)}| + \sum_{i=1}^j \lambda_i^{(k)} \geq \sum_{l=1}^{k-1} |\mu^{(l)}| + \sum_{i=1}^j \mu_i^{(k)}$$

for all j, k .

With each multipartition $\underline{\lambda}$, we can associate an \mathcal{H}_n -module $S^\underline{\lambda}$. Its concrete construction is explained in [DJM, (3.28)]. It is easy to see from the construction that it is free as an R -module. These modules are called **Specht modules**.

Each Specht module is naturally equipped with a bilinear form [DJM, (3.28)]. We set $D^\lambda = S^\lambda / \text{rad } S^\lambda$. It can be zero, but non-zero ones exhaust all simple \mathcal{H}_n -modules. We denote the projective cover of D^λ by P^λ .

We remark that Graham and Lehrer have introduced the notion of cellular algebras and have developed general theory for classifying simple modules by using "cell modules". In [GL], the cellular bases for the cell modules are given by Kazhdan-Lusztig bases. Here, different cellular bases are given, but the strategy to classify simple modules is the same. Hence we call the following parametrization the Graham-Lehrer parametrization.

Theorem 2.2 ([DJM, Theorem 3.30]) *Suppose that R is a field. Then,*

- (1) *Non-zero D^λ form a complete set of non-isomorphic simple \mathcal{H}_n -modules. Further, these modules are absolutely irreducible.*
- (2) *Let $\underline{\lambda}$ and $\underline{\mu}$ be multipartitions of n and suppose that $D^\underline{\mu} \neq 0$ and that $[S^\underline{\lambda} : D^\underline{\mu}] \neq 0$. Then we have $\underline{\lambda} \sqsupseteq \underline{\mu}$.*
- (3) $[S^\underline{\lambda} : D^\underline{\lambda}] = 1$.

Note that (2) is equivalent to the following (2').

- (2') *Let $\underline{\lambda}$ and $\underline{\mu}$ be multipartitions of n and suppose that $D^\underline{\mu} \neq 0$ and that $[P^\underline{\mu} : S^\underline{\lambda}] \neq 0$. Then we have $\underline{\lambda} \sqsupseteq \underline{\mu}$.*

It is obvious since we have $[P^\underline{\mu} : S^\underline{\lambda}] = \dim \text{Hom}_{\mathcal{H}_n}(P^\underline{\mu}, S^\underline{\lambda}) = [S^\underline{\lambda} : D^\underline{\mu}]$.

As is explained in [AM, 1.2], the classification of simple \mathcal{H}_n -modules is reduced to the classification in the case that u_1, \dots, u_m are powers of ζ^2 . This is a consequence of a result in [Vig1, 2.13] (see also [DM]). We can also assume that $\zeta^2 \neq 1$, since the case $\zeta^2 = 1$ is well understood. In the rest of the paper throughout, we assume that

$$u_i = \zeta^{2\gamma_i} \quad (i = 1, \dots, m), \quad \zeta^2 \neq 1$$

If ζ^2 is a primitive r th root of unity for a natural number r , γ_i take values in $\mathbb{Z}/r\mathbb{Z}$. Otherwise, these take values in \mathbb{Z} . We now recall the notion of **Kleshchev multipartitions** associated with $(\gamma_1, \dots, \gamma_m)$. To do this, we explain the notion of good nodes first.

We identify a multipartition $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ with the associated Young diagram, i.e. an m -tuple of the Young diagrams associated with $\lambda^{(1)}, \dots, \lambda^{(m)}$. Let x be a node on the Young diagram which is located on the a th row and the b th column of $\lambda^{(c)}$. If $u_c \zeta^{2(b-a)} = \zeta^{2i}$, we say that the node x has **residue** i (with respect to $\gamma = (\gamma_1, \dots, \gamma_m)$). We denote the residue by $r_\gamma(x)$. A node is called an **i -node** if its residue is i . Let $\underline{\lambda}$ and $\underline{\mu}$ be multipartitions. We first assume that $|\underline{\lambda}| + 1 = |\underline{\mu}|$, $r_\gamma(x) \equiv i$, and let x be the node $\underline{\mu}/\underline{\lambda}$. We then call x an **addable i -node** of $\underline{\lambda}$. If $|\underline{\lambda}| - 1 = |\underline{\mu}|$ and $r_\gamma(x) \equiv i$, we call $x = \underline{\lambda}/\underline{\mu}$ a **removable i -node** of $\underline{\lambda}$.

For each residue i , we have the notion of normal i -nodes and good i -nodes. To define these, we read addable and removable i -nodes of $\underline{\lambda}$ in the following way. We start with the first row of $\lambda^{(1)}$, and we read rows in $\lambda^{(1)}$ downward. We then move to the first row of $\lambda^{(2)}$, and repeat the same procedure. We continue the procedure to $\lambda^{(3)}, \dots, \lambda^{(m)}$. If we write A for an addable i -node, and similarly R for a removable i -node, we get a sequence of A and R . We then delete RA as many as possible. For example, if the sequence is $RRAAAARRRAARAR$, it ends up with $---AAR-----R$. The remaining removable i -nodes in this sequence are called **normal i -nodes**. The node corresponding to the leftmost R is called the **good i -node**. If x is a good i -node for some i , we simply say x is a **good node**. We can now define the set of Kleshchev multipartitions associated with $\gamma = (\gamma_1, \dots, \gamma_m)$.

Definition 2.3 *We declare that \emptyset is Kleshchev. Assume that we have already defined the set of Kleshchev multipartitions of size n .*

Let $\underline{\lambda}$ be a multipartition of size $n+1$. We say that $\underline{\lambda}$ is Kleshchev if and only if there is a good node x of $\underline{\lambda}$ such that $\underline{\mu} := \underline{\lambda} \setminus \{x\}$ is a Kleshchev multipartition.

We denote the set of Kleshchev multipartitions of size n by ${}^\gamma \mathcal{KP}_n$, and set ${}^\gamma \mathcal{KP} = \sqcup_{n \geq 0} {}^\gamma \mathcal{KP}_n$.

Theorem 2.4 ([AM, Theorem C]) *Suppose that ζ^2 and u_i satisfy the above condition. Then, the irreducible \mathcal{H}_n -modules are indexed by the set of Kleshchev multipartitions.*

Hence we have two parametrizations. One given in Theorem 2.4 and one given in Theorem 2.2. It is natural to ask, if these coincide. The main observation is the following conjecture, which will be proved in the last section. The conjecture was formulated by Mathas.

Conjecture[AM, 2.12] *These two parametrizations coincide. In particular, $D^{\underline{\lambda}} \neq 0$ if and only if $\underline{\lambda}$ is a Kleshchev multipartition.*

To prove this, we use certain Fock spaces over a quantum algebra. * In the next section, we recall necessary ingredients of these Fock spaces.

3 Fock space

Recall that the multiplicative order of ζ^2 is $r \geq 2$. We denote by U_v the quantum algebra of type $A_{r-1}^{(1)}$ if r is finite, and of type A_∞ if $r = \infty$. Let \mathcal{F}_v^γ be the combinatorial Fock space: it is a U_v -module, whose basis elements are indexed by the set of all multipartitions. We identify the basis elements with

*The idea to use such Fock spaces to study the modular representation theory of cyclotomic Hecke algebras first appeared in [A1], generalizing and verifying a conjecture of Lascoux, Leclerc and Thibon [LLT].

the multipartitions. The size of multipartitions naturally makes it into a graded vector space.

We consider the U_v -submodule \mathcal{M}_v^γ of \mathcal{F}_v^γ generated by \emptyset . It is isomorphic to an irreducible highest weight module with highest weight $\Lambda = \Lambda_{\gamma_1} + \dots + \Lambda_{\gamma_m}$, where Λ_i are fundamental weights. To describe its basis in a combinatorial way, we need the crystal graph theory of Kashiwara. In our particular setting, we can prove the following theorem using argument in [MM]. The theorem explains the representation theoretic meaning of Kleshchev multipartitions.

Theorem 3.1 ([AM, Theorem 2.9, Corollary 2.11]) *Let R_v be the localized ring of $\mathbb{Q}[v]$ with respect to the prime ideal (v) . We consider the R_v -lattice of \mathcal{F}_v^γ generated by all multipartitions, and denote it by \mathcal{L}_v^γ . We set $L(\Lambda) = \mathcal{L}_v^\gamma \cap \mathcal{M}_v^\gamma$, and $B(\Lambda) = \{\underline{\lambda} \text{ mod } v \mid \underline{\lambda} \in {}^\gamma \mathcal{K}P\}$. Then, $(L(\Lambda), B(\Lambda))$ is a (lower) crystal base of \mathcal{M}_v^γ in the sense of Kashiwara.*

To explain the U_v -module structure given to \mathcal{F}^γ , we first fix notations. Let $\underline{\lambda}$ be a multipartition and let x be a node on the associated Young diagram which is located on the a th row and b th column of $\lambda^{(c)}$. Then we say that a node is **above** x if it is on $\lambda^{(k)}$ for some $k < c$, or if it is on $\lambda^{(c)}$ and the row number is strictly smaller than a . We denote the set of addable (resp. removable) i -nodes of $\underline{\lambda}$ which are above x by $A_i^a(x)$ (resp. $R_i^a(x)$). In a similar way, we say that a node is **below** x if it is on $\lambda^{(k)}$ for some $k > c$, or if it is on $\lambda^{(c)}$ and the row number is strictly greater than a . We denote the set of addable (resp. removable) i -nodes of $\underline{\lambda}$ which are below x by $A_i^b(x)$ (resp. $R_i^b(x)$). The set of all addable (resp. removable) i -nodes of $\underline{\lambda}$ is denoted by $A_i(\underline{\lambda})$ (resp. $R_i(\underline{\lambda})$).

In the similar way, we define the notion that a node is **left** to x (resp. **right** to x). We denote the set of addable i -nodes which are left to x (resp. right to x) by $A_i^l(x)$ (resp. $A_i^r(x)$). The set of removable i -nodes which are left to x (resp. right to x) is denoted by $R_i^l(x)$ (resp. $R_i^r(x)$). We then set

$$N_i^a(x) = |A_i^a(x)| - |R_i^a(x)|, \quad N_i^b(x) = |A_i^b(x)| - |R_i^b(x)|$$

$$N_i(\underline{\lambda}) = |A_i(\underline{\lambda})| - |R_i(\underline{\lambda})|$$

$N_i^l(x)$ and $N_i^r(x)$ are similarly defined. Finally, we denote the number of all 0-nodes in $\underline{\lambda}$ by $N_d(\underline{\lambda})$. Then the U_v -module structure of \mathcal{F}_v^γ (called Hayashi action) is defined as follows.

$$e_i \underline{\lambda} = \sum_{r_\gamma(\underline{\lambda}/\underline{\mu}) \equiv i} v^{-N_i^a(\underline{\lambda}/\underline{\mu})} \underline{\mu}, \quad f_i \underline{\lambda} = \sum_{r_\gamma(\underline{\mu}/\underline{\lambda}) \equiv i} v^{N_i^b(\underline{\mu}/\underline{\lambda})} \underline{\mu}$$

$$v^{h_i} \underline{\lambda} = v^{N_i(\underline{\lambda})} \underline{\lambda}, \quad v^d \underline{\lambda} = v^{-N_d(\underline{\lambda})} \underline{\lambda}$$

Let $\mathcal{F}_v^{\gamma_i}$ ($i = 1, \dots, m$) be the combinatorial Fock spaces defined for the cases that $m=1$ and $\gamma = \gamma_i$. The U_v -module structure on these spaces are defined by the same formula given above. Let Δ' be the comultiplication defined by

$$\Delta'(e_i) = v^{-h_i} \otimes e_i + e_i \otimes 1, \quad \Delta'(f_i) = 1 \otimes f_i + f_i \otimes v^{h_i}$$

If we identify \mathcal{F}_v^γ with $\mathcal{F}_v^{\gamma_1} \otimes \cdots \otimes \mathcal{F}_v^{\gamma_m}$, the representation on \mathcal{F}_v^γ coincides with the tensor product representation defined with respect to Δ' .

Let $-\gamma' := (-\gamma_m, \dots, -\gamma_1)$. Namely, we write $r_{-\gamma'}(x) \equiv i$ if a node x on the a th row and the b th column of the $\lambda^{(c)}$ satisfies $u_{m+1-c}^{-1} \zeta^{2(b-a)} = \zeta^{2i}$.

For each partition λ we denote its transpose by λ' . For a multipartition $\underline{\lambda}$, we denote $(\lambda^{(m)'}', \dots, \lambda^{(1)'}')$ by $\underline{\lambda}'$ and call it the **flip transpose** of $\underline{\lambda}$. Similarly, we denote $(\lambda^{(1)'}', \dots, \lambda^{(m)'}')$ by $\underline{\lambda}^T$ and call it the **transpose** of $\underline{\lambda}$.

Let $\sigma : \mathcal{F}_v^{-\gamma'} \rightarrow \mathcal{F}_v^\gamma$ be a linear map which maps $\underline{\lambda}$ to $\underline{\lambda}'$. Then the coproduct on $\mathcal{F}_v^{-\gamma'}$ coincides with Kashiwara's, and the action coincides with [LLT]. Hence the R_v -lattice generated by $\underline{\lambda}$ is a crystal lattice in \mathcal{F}_v^γ .

Let $\xi : \mathcal{F}_v^\gamma \rightarrow \mathcal{F}_{v-1}^{-\gamma}$ be a semilinear map which sends $\underline{\lambda}$ to $\underline{\lambda}^T$. Then we have a representation which is compatible with Lusztig's coproduct. The space $\mathcal{F}_{v-1}^{-\gamma}$ is the same space as $\mathcal{F}_v^{-\gamma}$, but to stress that the crystal base here is a so-called "basis at $v = \infty$ " in the sense of Lusztig, and not the one generated over R_v by $\underline{\lambda}$, we adopt the different notation. The action on $\mathcal{F}_{v-1}^{-\gamma}$ is as follows. We also call it Hayashi action.

$$e_i \underline{\lambda} = \sum_{r_\gamma(\underline{\lambda}/\underline{\mu}) \equiv i} v^{N_i^l(\underline{\lambda}/\underline{\mu})} \underline{\mu}, \quad f_i \underline{\lambda} = \sum_{r_\gamma(\underline{\mu}/\underline{\lambda}) \equiv i} v^{-N_i^r(\underline{\mu}/\underline{\lambda})} \underline{\mu}$$

$$v^{h_i} \underline{\lambda} = v^{N_i(\underline{\lambda})} \underline{\lambda}, \quad v^d \underline{\lambda} = v^{-N_d(\underline{\lambda})} \underline{\lambda}$$

In the rest of paper, we exclusively work with $\mathcal{F}_{v-1}^{-\gamma}$.

4 The proof of the conjecture

We first interpret the conjecture into a problem about canonical bases on Fock spaces. To do this, we use the direct sum of the Grothendieck groups of projective \mathcal{H}_n -modules. We always assume that the coefficients are extended to the field of rational numbers. If \mathcal{H}_n is semisimple, all $S^\underline{\lambda}$ are irreducible, and we identify the direct sum with $\mathcal{F}_{v=1}^\gamma$, which is by definition a based \mathbb{Q} -vector space whose basis elements are indexed by multipartitions, and nodes of multipartitions are given residues. If \mathcal{H}_n is not semisimple, we have a proper subspace of $\mathcal{F}_{v=1}^\gamma$ by lifting idempotents argument. It is proved in [A1] that it coincides with $\mathcal{M}_{v=1}^\gamma$.

Recall that simple modules are obtained as factor modules of Specht modules. To distinguish between simple modules over different base rings, we write $D_R^\underline{\lambda}$ when the base ring is R . Let (K, R, F) be a modular system. We assume that there is an invertible element $\zeta \in R$ such that its multiplicative order in K and F is the same. Then $D_K^\underline{\lambda}$ is obtained from $D_R^\underline{\lambda}$ by extension of coefficients, and $D_F^\underline{\lambda}$ is obtained from $D_R^\underline{\lambda}$ by taking the unique simple factor module of $D_R^\underline{\lambda} \otimes F$. The proof of Theorem 2.4 implies that these give the correspondence between simple modules over fields of positive characteristics and fields of characteristic 0, and $D_F^\underline{\lambda} \neq 0$ if and only if $D_K^\underline{\lambda} \neq 0$. Further, still assuming that the multiplicative order is the same, the proof given in [AM] also shows that

$D_K^\lambda \neq 0$ if and only if $D_{\mathbb{C}}^\lambda \neq 0$. In particular, to know which D^λ are non-zero, it is enough to consider the case that the base field is \mathbb{C} .

Now assume that we are in the case that the base field is \mathbb{C} . We identify the direct sum of the Grothendieck groups of projective \mathcal{H}_n -modules with $\mathcal{M}_{v=1}^\gamma$ as before. The main theorem in [A1] asserts that the canonical basis evaluated at $v=1$ consists of indecomposable projective \mathcal{H}_n -modules ($n = 0, 1, \dots$). Hence we have a bijection between canonical basis elements of \mathcal{M}_v^γ and indecomposable projective \mathcal{H}_n -modules P^λ for various n , and thus a bijection between canonical basis elements of \mathcal{M}_v^γ and simple \mathcal{H}_n -modules D^λ for various n .

It is known that the canonical basis gives a crystal base of \mathcal{M}_v^γ [GL], which is unique up to scalar [Ka]. More precisely, the crystal lattice $L(\Lambda)$ is the R_v -lattice generated by the canonical basis elements, and $B(\lambda)$ consists of the canonical basis elements modulo v . Then Theorem 3.1 asserts that with each canonical basis element $G(b)$, we can uniquely associate a multipartition $\underline{\nu} \in {}^\gamma \mathcal{KP}$. To summarize, we have the following.

For each non-zero D^λ , there exists a unique canonical basis element $G(b)$ such that we have $G(b)_{v=1} = P^\lambda$ and $G(b) \equiv \underline{\nu} \bmod vL(\Lambda)$.

If $\underline{\nu} = \underline{\lambda}$ holds in general, then the following lemma proves the conjecture. The dominance order on $\mathcal{F}_{v=1}^{-\gamma}$ is defined by reading columns from left to right. If we read the columns from right to left, we have the reversed dominance order. We denote it by $\underline{\lambda} \geq \underline{\mu}$.

Lemma 4.1 *Assume that for every canonical basis element $G(b)$ in \mathcal{F}_v^γ , there exists a unique maximal element among the multipartitions appearing in $\xi(G(b))$ with respect to the reversed dominance order, and assume that it has coefficient 1. Then we have that $D^\lambda \neq 0$ if and only if $\underline{\lambda}$ is a Kleshchev multipartition.*

(Proof) Recall that $\mathcal{M}_{v=1}^\gamma$, the sum of Grothendieck groups of projective \mathcal{H}_n -modules, is embedded into $\mathcal{F}_{v=1}^\gamma$ by sending S^λ to $\underline{\lambda}$. Hence, Theorem 2.2 implies that $\xi(P^\lambda)$ has the form $\xi(P^\lambda) = \underline{\lambda}^T + \sum_{\underline{\mu} < \underline{\lambda}^T} c_{\underline{\mu}} \underline{\mu}$. In particular, among multipartitions appearing in $\xi(P^\lambda)$, $\underline{\lambda}^T$ is the maximal element with respect to the dominance order.

On the other hand, Theorem 2.4 implies that there exists a canonical basis element $G(b)$ such that $P^\lambda = G(b)_{v=1}$. We apply the assumption to $\xi(G(b))$. Then multipartitions appearing in $\xi(G(b))$ has a unique maximal element with coefficient 1. Since it is a canonical basis element and the coefficient is 1, it must be the transpose of a Kleshchev multipartition, say $\underline{\nu}$. We specialize $\xi(G(b))$ to $v = 1$. Note that $\underline{\nu}$ does not vanish. Since both $\underline{\lambda}^T$ and $\underline{\nu}^T$ are maximal elements, we have $\underline{\nu} = \underline{\lambda}$. Hence the two parametrizations given in Theorem 2.2 and Theorem 2.4 coincide. ■

We say that $\tilde{\gamma} = (\tilde{\gamma}_1, \dots, \tilde{\gamma}_m)$ is a **lift** of $\gamma = (\gamma_1, \dots, \gamma_m)$ if $\tilde{\gamma}_i \bmod r = \gamma_i$ for all i . If $r = \infty$, we set $\tilde{\gamma} = \gamma$. For each $\tilde{\gamma}$, we have $\mathcal{F}_{v=1}^{-\tilde{\gamma}}$, and it is a module over the quantum algebra of type A_∞ . It will play a main role in the following.

In [TU], Takemura and Uglov constructed higher level Fock spaces by generalizing [KMS, Proposition 1.4]. Let $\{u_i\}_{i \in \mathbb{Z}}$ be the basis vectors of an infinite dimensional space. More precisely, the space is originally $\mathbb{Q}(v)^r \otimes \mathbb{Q}(v)^m[z, z^{-1}]$, and if we denote the basis elements by $e_a \otimes e_b z^N$, we identify u_i with $e_a \otimes e_b z^N$ through $i = a + r(b - 1 - mN)$ as in [U]. Note that this identification is different from that in [TU] since the evaluation representation for $U'_v(\hat{\mathfrak{sl}}_m)$ taken in [TU] is different from that in [U]. This space is naturally a $U'_v(\hat{\mathfrak{sl}}_r) \otimes U'_v(\hat{\mathfrak{sl}}_m)$ -module. The semi-infinite wedges of the form $u_I = u_{i_1} \wedge u_{i_2} \wedge \dots$ such that $i_k = c - k + 1$ for all $k >> 0$ are said to have charge c . The space of semi-infinite wedges of charge c is denoted by \mathcal{F}_c . To make \mathcal{F}_c into a $U_v(\hat{\mathfrak{sl}}_r) \otimes U_v(\hat{\mathfrak{sl}}_m)$ -module, we use the following coproducts.

$$\Delta^{(l)}(f_{\bar{i}}) = f_{\bar{i}} \otimes 1 + v^{-h_{\bar{i}}} \otimes f_{\bar{i}}$$

$$\Delta^{(r)}(f_{\bar{i}}) = f_{\bar{i}} \otimes 1 + v^{h_{\bar{i}}} \otimes f_{\bar{i}}$$

Note that $\Delta^{(l)}$ (resp. $\Delta^{(r)}$) is obtained from Lusztig's coproduct (resp. Δ') by reversing the order of the tensor product. The only reason we use them here is that we are more familiar with semi-infinite wedges which are infinite to the right. $\Delta^{(l)}$ behaves well for the bases at $v = \infty$. On the other hand, $\Delta^{(r)}$ behaves well for Kashiwara's lower crystal bases. Thus we are mostly concerned with $\Delta^{(l)}$ to work with $\mathcal{F}_{v-1}^{-\tilde{\gamma}}$.

If i_k are in descending order, they are called **normally ordered** wedges. The straightening laws are explained in [U, (Ri)(Rii)(Riii)(Riv)]. The normally ordered semi-infinite wedges of charge c form a basis of \mathcal{F}_c [TU, Proposition 4.1]. For a normally ordered wedge, we locate them on an abacus with rm runners. On each runner, larger numbers appear in upper location, and the row containing 1 is read $1, \dots, rm$ from left to right. We divide the set of these runners into m blocks. Then we have m abacuses each of which has r runners. By reading i_k 's in each block, we have m semi-infinite wedges. We now assume that these are of the form $u_I^{(k)} := u_{j_1^{(k)}} \wedge u_{j_2^{(k)}} \wedge \dots$ such that $j_i^{(k)} = -\tilde{\gamma}_k - i + 1$ for all k and $i >> 0$. We then identify $u_I^{(k)}$ with a multipartition $\lambda^{(k)}$ by $j_i^{(k)} = -\tilde{\gamma}_k + \lambda^{(k)}_i - i + 1$. We identify $\mathcal{F}_{v-1}^{-\tilde{\gamma}}$ with the subspace of \mathcal{F}_c ($c = -\sum \tilde{\gamma}_k$) spanned by the wedges u_I whose $u_I^{(k)}$ have this form. This correspondence from normally ordered wedges to multipartitions is compatible with the action of $U_v(\hat{\mathfrak{sl}}_r)$. (If we consider the usual abacus with r runners, it is compatible with $U_v(\hat{\mathfrak{sl}}_m)$ -action.) More precisely, for each n , we take $\tilde{\gamma}$ such that $-\tilde{\gamma}_k \ll -\tilde{\gamma}_{k+1}$ for all k . Then the action of $U_v(\hat{\mathfrak{sl}}_r)$ on the multipartitions of size less than n coincides with the action given to $\mathcal{F}_{v-1}^{-\tilde{\gamma}}$. This follows from the definition of the coproduct $\Delta^{(l)}$.

We are now in a position to introduce a bar operation on the space of semi-infinite wedges as in [U, 3.1]. The definition is identical to the definition of the bar operation on level one modules introduced in [LT1, Proposition 3.1]. The welldefinedness for level one modules is given in [LT2, 5.1-5.9]. The same proof works for the semi-infinite wedges considered here. We also have that $f_{\bar{i}}$

commutes with the bar operation, and that the bar operation preserves the size of multipartitions.

We state the properties of the bar operation due to Uglov. For level one modules, these are stated in [LT1, Theorem 3.2, Theorem 3.3]. (The proof is given in [LT2, 7.1-7.4].)

Theorem 4.2 ([U, Theorem 3.2, Theorem 3.3])

- (1) *The bar operation preserves $\mathcal{F}_{v^{-1}}^{-\tilde{\gamma}}$.*
- (2) *$\overline{f_i \lambda} = f_{\bar{i}} \bar{\lambda}$, and $\overline{\emptyset} = \emptyset$. In particular, the bar operation is an extension of the bar operation defined on $\mathcal{M}_{v^{-1}}^{-\gamma}$.*
- (3) *For each n , we take $\tilde{\gamma}$ as before. Then for multipartitions of size less than n , we have that $\bar{\lambda}$ has the form $\lambda + \sum_{\underline{\mu} < \lambda} \alpha_{\lambda, \underline{\mu}}(v) \underline{\mu}$.*

The validity of $\overline{\emptyset} = \emptyset$ comes from the facts that the bar operation preserves the subspace $\mathcal{F}_{v^{-1}}^{-\tilde{\gamma}}$, and \emptyset is the unique multipartiton of the minimum size. The straightening laws show the unitriangularity of the bar operation. Note that the dominance order in [U] corresponds to the reversed dominance order here. This triangularity also gives an algorithm to compute canonical basis on higher level modules. Thus it also computes decomposition numbers of cyclotomic Hecke algebras of type $G(m, 1, n)$ over the field of complex numbers [A1].

Since this theorem gives the required property of the canonical basis elements in question, we have reached the following theorem, which verifies the conjecture.

Theorem 4.3 $D^\lambda \neq 0$ if and only if λ is a Kleshchev multipartition.

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